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Formulating Analytical Solution of Network ODE Systems Based on Input Excitations

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Abstract

The concepts of graph theory are applied to model and analyze dynamics of computer networks, biochemical networks and, semantics of social networks. The analysis of dynamics of complex networks is important in order to determine the stability and performance of networked systems. The analysis of non-stationary and nonlinear complex networks requires the applications of ordinary differential equations (ODE). However, the process of resolving input excitation to the dynamic non-stationary networks is difficult without involving external functions. This paper proposes an analytical formulation for generating solutions of nonlinear network ODE systems with functional decomposition. Furthermore, the input excitations are analytically resolved in linearized dynamic networks. The stability condition of dynamic networks is determined. The proposed analytical framework is generalized in nature and does not require any domain or range constraints.

Keywords

Computer Networks, Convergent Functions, Dynamic Networks, Ordinary Differential Equations

1. Introduction

The graph theoretic models of complex network systems are often applied to analyze computer networks, biochemical networks and semantics of social networks. In general, the complex networks can be classified into stationary networks and non-stationary networks. The stationary networks are having static structures and, the flow control in the networks is one of the main concerns. The wired computer networks are the examples of stationary networks. However, the dynamic networks are having polymorphic structures and, the stability of the network dynamics is one of the main concerns. The wireless networks of computers as well as social networks are prime examples of non-stationary and highly dynamic networked systems [1]. The stationary networks are often modeled by employing graph theory and, the flow controls in stationary networks are determined through graph algorithms. However, it is noted that the stationary network flows often indicate the existence of underlying stochastic elements [2]. Although the wired networks of computers are stationary graphs in nature, however the data traffic in stationary computer networks follows periodic oscillatory processes [3]. The modeling and analysis of dynamic networks are conducted either by following non-stationary process models or by employing the randomized Boolean functional networks [4].

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In dynamic networked systems, the transformations of Boolean functional networks are utilized for modeling [5]. For example, the modeling and analysis of wireless computer networks are performed using Markov models [6]. In general, the complex dynamic networks are modeled by employing ordinary differential equations (ODE) having nonlinear forms along with input excitations. Often, the ODE models of dynamic networks are linearizable and, corresponding input excitations determine the behaviour of network systems [3]. The formulation of analytical solution of nonlinear ODE models of non-stationary networks is required in order to understand the general dynamics of network systems as well as stability. Moreover, the input excitation to such dynamic network systems is required to be resolved independent of functions of general homogeneous solutions.

1.1 Motivation

The complex non-stationary networks are often modeled by using nonlinear non-homogeneous ODE systems. In general, the complex dynamic networks are driven by input excitations. The input excitations coupled with the dynamics of nonlinear ODE determine the stability of non-stationary network systems. The construction of analytical solutions of nonlinear non-homogeneous network ODE is required in order to compute the general dynamics of the systems. Moreover, the input excitation is required to be resolved independent of homogeneous solutions in case the order of the system is increased. This paper proposes the analytical solution of dynamic network ODE in general form under decomposition and, the input excitation is resolved if the order of the dynamic network systems is increased. The stability conditions of the networks are also determined in such higher order dynamic network systems. The main contributions of this paper are as follows:

- Constructing an analytical formulation of nonlinear non-stationary network ODE systems using functional decomposition.
- Deriving an integrated analytical solution of linearized ODE resolving input excitation to networks independent of external functions.
- Determining the stability conditions of non-stationary networks.

Rest of the paper is organized as follows. The second section of this paper presents related work. The third section of this paper describes proposed analytical model of network ODE systems. The fourth section depicts the experimental evaluations of solution. Finally, fifth section concludes the paper.

2. Related Work

The modeling and analysis of computer networks and other physical networked systems employ graph theoretic approaches along with queuing theory [7,8]. However, the formulation of network dynamics in analytic forms using ODE is an effective approach [9]. The non-stationary dynamics of computer networks are modeled by using non-linear ODE in combination of queuing theory [1]. However, the dynamical behaviour of non-stationary computer networks can be modeled by using first order ODE along with Ateb-functions [3]. The dynamics of computer networks exhibit the properties of discrete dynamical systems and it can be modeled by employing the formalisms of DEVS (Discrete Events Systems Specifications) [2]. Furthermore, the stochastic nature of computer networks is analyzed by applying stochastic DEVS. The stochastic DEVS model is constructed based on quantized state systems [2]. The network modeling and analysis by using quantized state systems require ODE systems incorporating approximations.

The biological/biochemical networks are formulated by using systems of ODE and Boolean functions [4,10]. The analysis of a randomized large-scale Boolean network requires the network model formulated by employing ODE [4]. The large-scale complex networks exhibit transient and steady-state dynamics, which can be modeled by using ODE and Euler-like transformations [5,11]. In another approach, the epidemiological model is employed to analyze information diffusion in large-scale networks [12,13]. Moreover, the information diffusion model for social networks is formulated by using logistic equation based on ODE [14]. The transient behaviour of wireless sensor networks is modeled by using the system of ODE having first order dynamics [6]. The wireless networks are prone to malware attacks and, can be modeled by using explicit network diffusion [15]. The finding of accurate and exact solutions of a system of ODE requires large computing capacities offered by high-performance distributed computing systems [16-18]. It is proposed that, the decomposed solvers for higher order ODE can be effectively implemented by using radial basis functions [19]. The main challenge in computational analysis of ODE systems is that, the decomposed solvers generate large data sets requiring extensive storage space. Thus, an analytical model is required to analyze the ODE systems with reduced complexity having appropriate linearization.

3. Analytical Models and Solutions

3.1 Nonlinear Networks ODE Models

This section considers a first-order nonlinear non-homogeneous ODE model representing the generalized non-stationary networks. The model of dynamic networks is represented by following equation in general form,

$$Dy + a(x, y)y = b(x, y). \tag{1}$$

If the coefficients are separable as, $a(x, y)/b(x, y) = a_1(x)a_2(y)/b_1(x)b_2(y)$ and, there is an arbitrary function $g(x) \neq 0$, then the equation can be transformed into following form considering $g(x)$ be integrating factor (k_1 is a constant),

$$\begin{aligned} &D(y\alpha) = b_2(y)^{-1} Dy, \\ &g(x) = k_1 e^{\int a_1(x)dx}, k_1 \in (-\infty, +\infty), \\ &\alpha(y) = a_2(y)/b_2(y) \end{aligned} \tag{2}$$

It immediately validates the following equation considering $[\forall y \in (+\infty, -\infty)] \Rightarrow [-\infty < \alpha(y) < +\infty]$,

$$[b_2(y)^{-1} - \alpha(y)]Dy = yD\alpha. \tag{3}$$

Thus, the formulation of an equation in separable form of nonlinear non-homogeneous ODE of networks can be constructed satisfying the condition, $D[g(x)\alpha(y)y] = b_1(x)g(x)$. However, from this condition and, from Eq. (3), one can further derive as (k_2 is a constant),

$$\alpha(y)\left[[b_2(y)^{-1} - \alpha(y)]/D\alpha\right]Dy = g(x)^{-1}\left(\int b_1(x)g(x)dx + k_2\right), \quad k_2 \in (-\infty, +\infty) \tag{4}$$

It can be concluded that, a general analytical solution of dynamic networks ODE can be formulated in the following functional form (k_3 is a constant),

$$\begin{aligned} F(y) &= \int g(x)^{-1}\left[\int b_1(x)g(x)dx + k_2\right]dx + k_3, \\ DF &= \left(D\left(a_2(y)b_2(y)^{-1}\right)\right)^{-1}\left([a_2(y) - a_2(y)^2]/b_2(y)^2\right), \\ k_3 &\in (-\infty, +\infty) \end{aligned} \tag{5}$$

The above equation represents analytical solution as well as the condition to be maintained while generating the solution intervals of non-stationary networks ODE models. A more applicable and direct method is to enhance the order of non-homogeneous ODE of non-stationary networks and to consider the input excitation to the network systems in one-dimension as described in next section.

3.2 Input Excitation in Linearized Networks

This section considers the higher-order non-stationary networks in linearized non-homogeneous form. The non-stationary network system is comprised of 2nd order linear non-homogeneous ODE having general form given as,

$$D^2y + a(x)Dy + b(x)y = i(x). \tag{6}$$

The coefficients of ODE are considered to be varying and the network is controlled by instantaneous input excitation, $i(x)$. Let the analytical solution component be $p(x)$ for the non-homogeneous ODE in general form given in Eq. (6). The network systems are considered under excitation-control having a functional factor, $h(x)$. This results in the excitation dependent solution having the following form,

$$p(x) = h(x)i(x). \tag{7}$$

Thus, the dynamics of the network systems due to varying input can be computed as,

$$D^2p = i(x)D^2h + h(x)D^2i + 2(Di)(Dh). \tag{8}$$

If the variations of coefficients of Eq. (6) are tightly coupled to the solution factor $h(x)$ controlling the instantaneous network dynamics, then the following linear combination should be satisfied for the corresponding network systems,

$$i(x)Dh + h(x)Di = -h(x)i(x)\left(\frac{b(x)}{a(x)}\right). \tag{9}$$

Moreover, the particular solution should satisfy ODE in original form as given in Eq. (10),

$$D^2 p + a(x)Dp + b(x)p(x) = i(x). \tag{10}$$

The respective equation of input excitation-controlled network ODE satisfying the solution takes following form,

$$i(x)D^2 h + h(x)D^2 i + 2(Di)(Dh) = i(x). \tag{11}$$

The general solution in terms of input excitation for a network ODE system can be derived from Eqs. (9) and (11) as,

$$\begin{aligned} \beta(x) &= b(x)/a(x), \\ \beta_g(x) &= -\int \beta(x)dx \end{aligned} \tag{12}$$

However, Eqs. (8), (11) and (12) lead to the condition in the network dynamics of the respective system restricting the behaviour of $h(x)$ such that (k is a constant),

$$\begin{aligned} h(x) &= \left[\frac{1}{\beta(x)^2 - D\beta}\right], \\ h(x)i(x) &= ke^{\beta_g(x)}, k \in (-\infty, +\infty) \end{aligned} \tag{13}$$

Furthermore, the corresponding network system exhibits the following control-dynamics as illustrated in Eq. (14),

$$h(x)^{-1} Dh + i(x)^{-1} Di = -\beta(x). \tag{14}$$

It can be observed from Eqs. (13) and (14) representing the characteristics that, the functional factor $h(x)$ controlling the network dynamics is governed by the coefficients of linearized ODE and, the input excitation to the dynamic networks is resolved independent of general homogeneous solutions involving any external function. The stability of the network systems can be derived by following the nature of Eq. (13).

3.3 Characteristics and Stability

The maintenance of stability of the dynamic non-stationary networks considering different types of input excitations is an important factor. The characteristics of control function of non-stationary network systems can be determined from the analytical solutions and associated conditions. The stability of non-stationary network systems is highly dependent on the characteristics function $h(x)$ and its interplay with input excitations. The two possible cases may arise such as, (a) $\exists c \in (-\infty, +\infty) : \lim_{x \to +\infty} \beta(x) = c$, which signifies that ratio of coefficients of networks ODE is convergent and, (b) $\forall c \in (-\infty, +\infty) : \lim_{x \to +\infty} \beta(x) \neq c$, which implies that the corresponding ratio is divergent in nature. It is important to note that, a divergent ratio does not immediately imply that the response of the networks would be unstable given a series of input excitations. On the other hand, a converging ratio does not imply that the response of non-stationary networks would be always stable under any arbitrary input excitations. The determination of stability is controlled by a properly conditioned characteristic function for a particular input excitation function.

4. Computational Evaluations

The computational evaluations are carried out in order to understand the interplay of characteristic function, coefficient ratio of non-stationary networks ODE and, different input excitations in continuum. In order to maintain applicability of the proposed networks analysis in practice, the input excitation is considered to be always positive having varying functional characteristics.

4.1 Classes of Input Excitations

The dynamics of network systems are computed with various input excitations having varying characteristics. The input excitations are classified into five categories such as (1) monotonically decreasing input excitations, (2) monotonically increasing excitations, (3) bounded oscillatory excitations, (4) damped oscillatory input excitations and, (5) unbounded and undamped oscillatory input excitations. The profiles of input excitations to networks are depicted in Figs. 1–5.

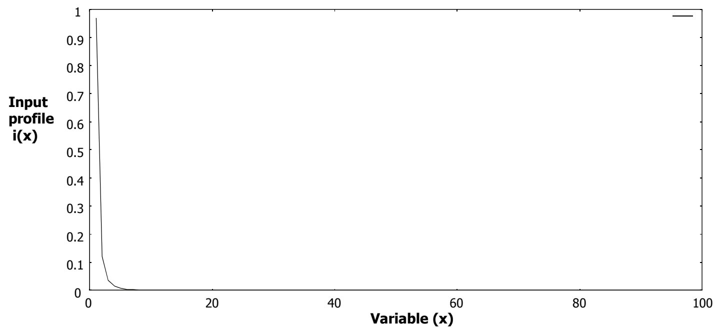

Fig. 1. Monotonically decreasing input profile.

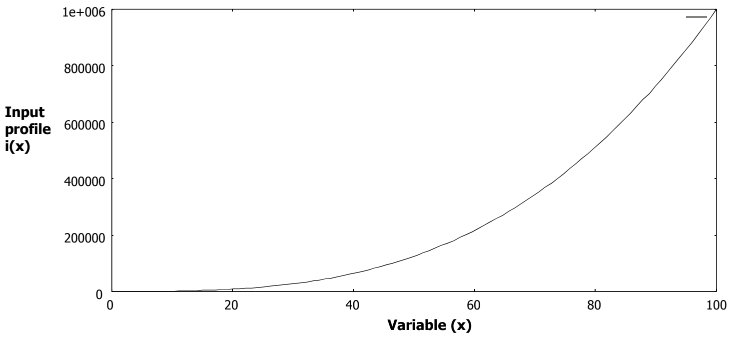

Fig. 2. Monotonically increasing input profile.

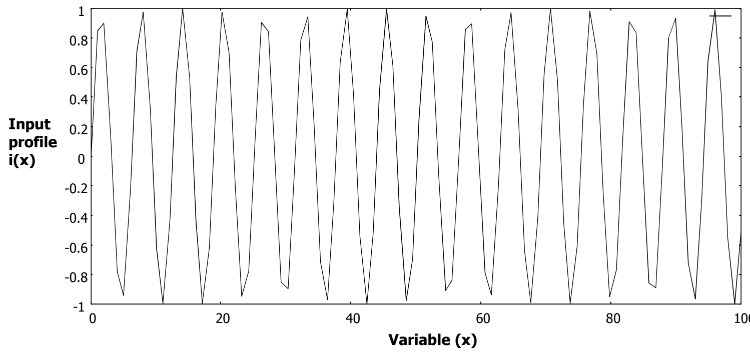

Fig. 3. Oscillatory input profile.

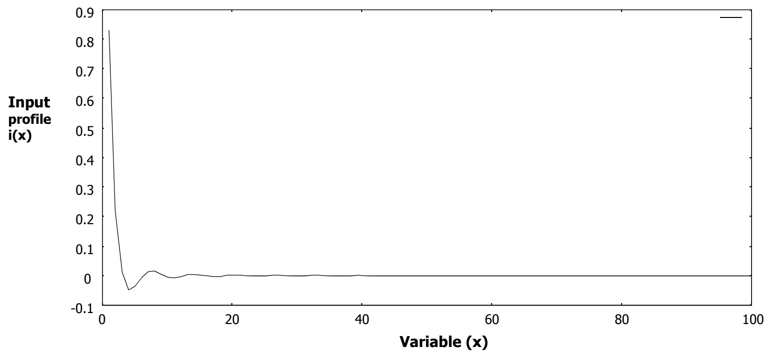

Fig. 4. Damped oscillatory input profile.

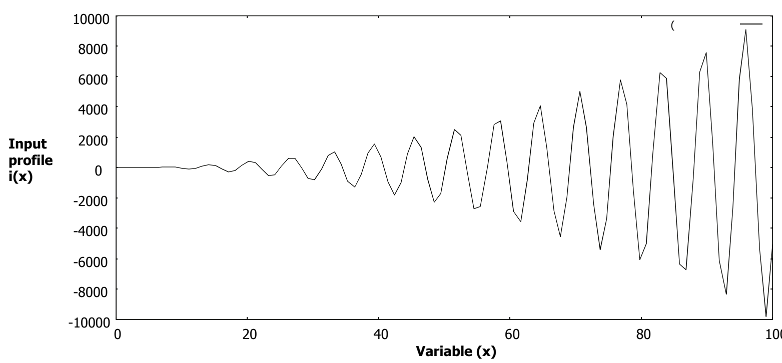

Fig. 5. Undamped oscillatory input profile.

The computational evaluations are performed with network excitations and varying coefficient ratios. The computational evaluations are conducted following two different cases depending upon the convergent property of coefficients.

4.2 Case I: Convergent Coefficient Ratio with Positive Input Excitation

In this case, the convergent coefficient ratio is considered and a set of different input excitations are applied to the computational model. The input excitations are categorized into five classes such as, monotonically decreasing input excitations, monotonically increasing input excitations and, periodic excitations in normal, damped and undamped forms. The profiles of corresponding network characteristic functions are illustrated in Figs. 6–10.

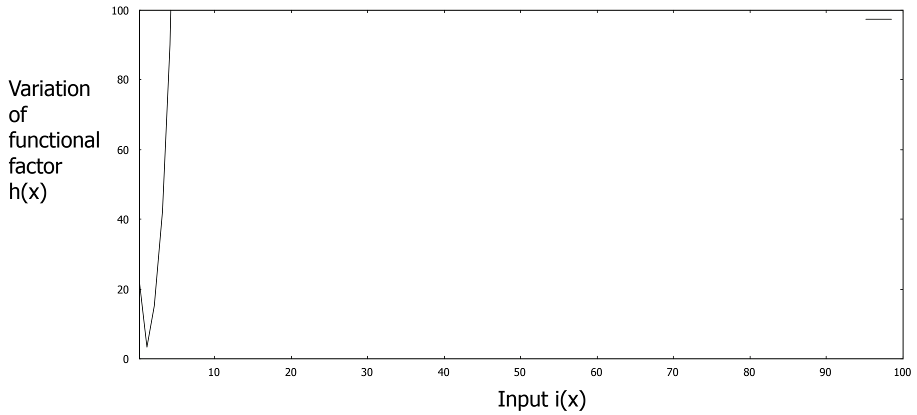

Fig. 6. Profile of network characteristic function for monotonically decreasing input.

It is observable from Fig. 6 that, the profile of network characteristic function is having narrow radius of convergence. The main reason is that, the input excitation is monotonically decreasing in nature having convergence. Thus, a narrow radius of convergence in corresponding characteristic function is suitable to maintain stability of non-stationary network systems.

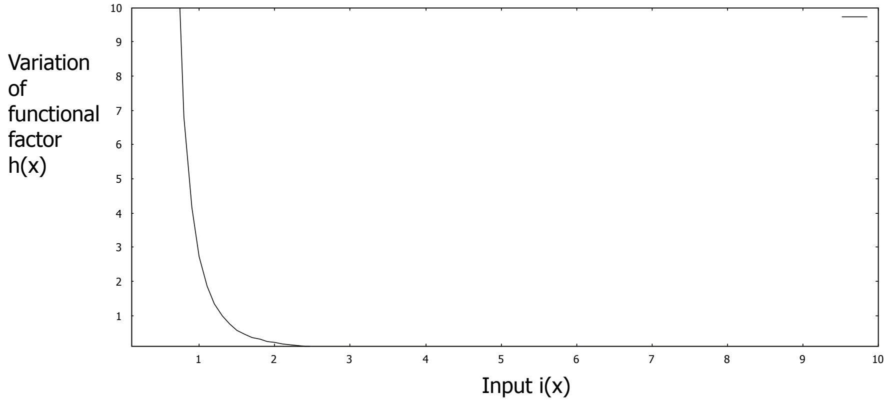

Fig. 7. Profile of network characteristic function for monotonically increasing input.

However, Fig. 7 illustrates that, in order to maintain stability a sharp and non-linear converging characteristic function is required if the input excitation is diverging in nature.

Fig. 8 illustrates that, if the input excitation is bounded and periodic in nature, then the corresponding network characteristic function of non-stationary networks requires to be bounded. However, if the periodic input is damped, then the network characteristic function may have two convergent and divergent regions as illustrated in Fig. 9. The radius of convergence is existed in a narrow range due to damping in input excitation to the networks.

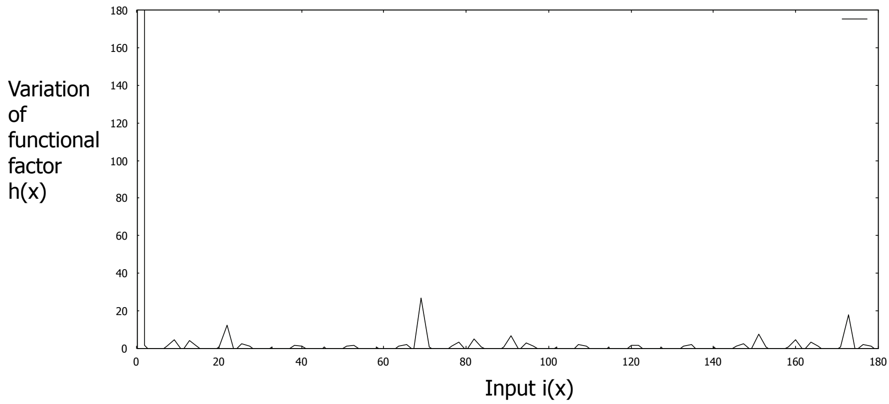

Fig. 8. Profile of network characteristic function for periodic input excitation.

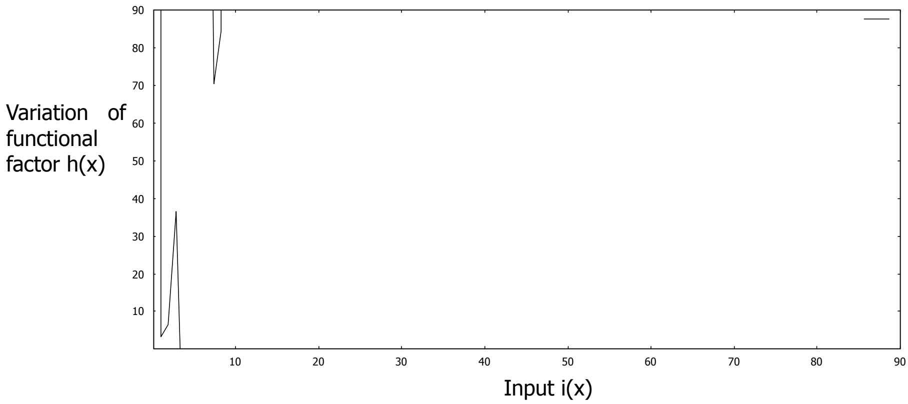

Fig. 9. Profile of network characteristic function for damped periodic input excitation.

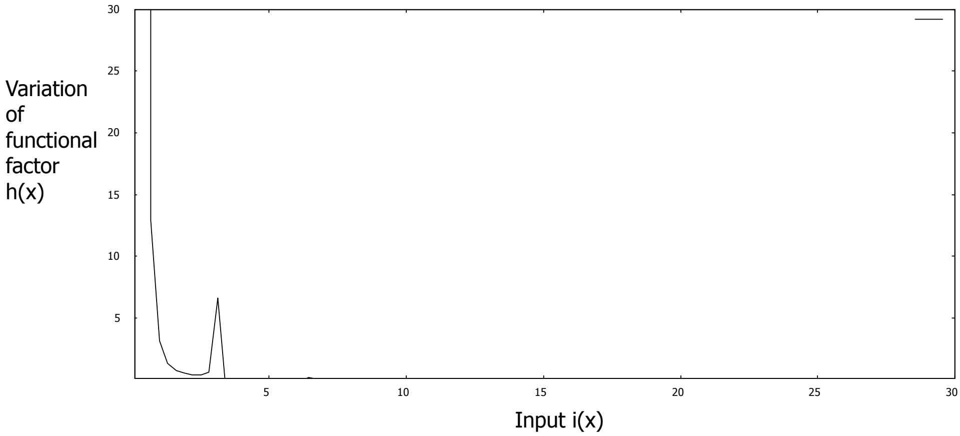

Fig. 10. Profile of network characteristic function for undamped periodic input excitation.

Furthermore, the profile of network characteristic function depicts a strong convergence property if the input excitation is undamped periodic in nature. The resulting profile is illustrated in Fig. 10. The surface map of response of the non-stationary network systems is depicted in Fig. 11. It is observable that, the response of networks is stable in all conditions if appropriate characteristic functions are applied. The convergence of responses to a surface of stability is rapid in nature.

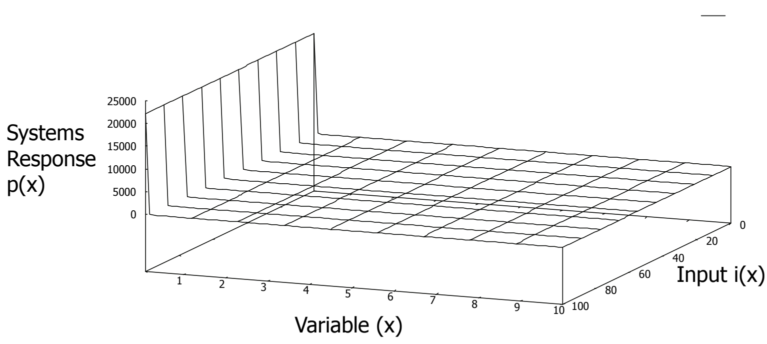

Fig. 11. Surface map of network response for convergent coefficient ratio.

4.3 Case II: Divergent Coefficient Ratio with Positive Input Excitation

In this case, the coefficient ratio of non-stationary network ODE systems is considered to be divergent in nature. The computational evaluations are conducted under five sets of input excitations having different characteristics (as earlier). The profiles of corresponding characteristic functions are illustrated in Figs. 12–16. Fig. 12 illustrates that, the network characteristic function exhibits symmetric profile with narrow radius of convergence if the input excitation is monotonically decreasing. However, a relatively sharp and non-linear convergent profile is depicted by network characteristic function, as illustrated in Fig. 13, if the input to network is diverging in nature. A similar effect is observable if the input excitation is changed to be periodic and bounded as depicted in Fig. 14. The main reason is that, the coefficient ratio of network ODE systems is diverging in nature. If the networks inputs are changed to damped and undamped periodic excitations, then the profiles of corresponding characteristic functions exhibit convergent profiles as illustrated in Figs. 15 and 16.

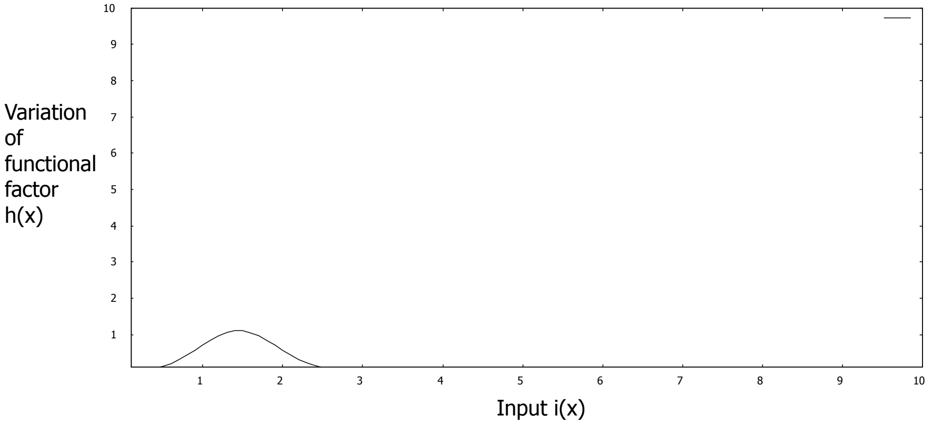

Fig. 12. Profile of network characteristic function for monotonically decreasing input.

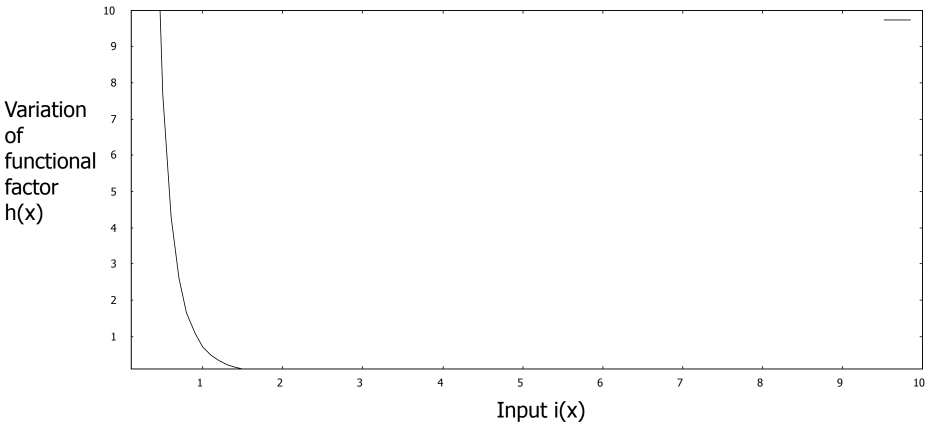

Fig. 13. Profile of network characteristic function for monotonically increasing input.

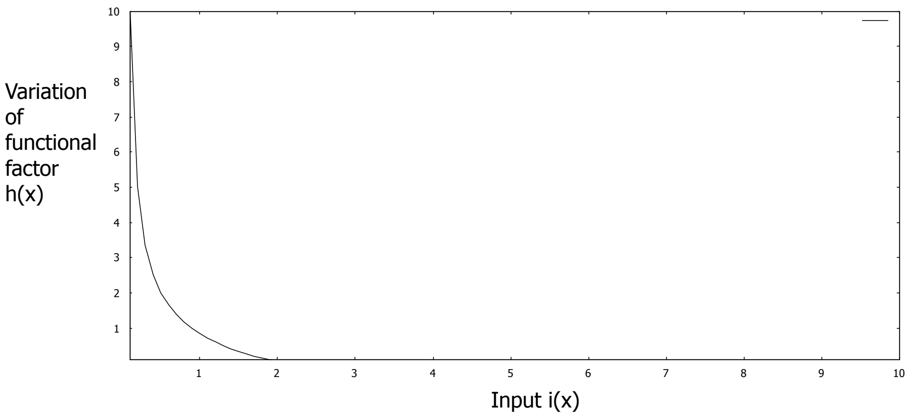

Fig. 14. Profile of network characteristic function for periodic input excitation.

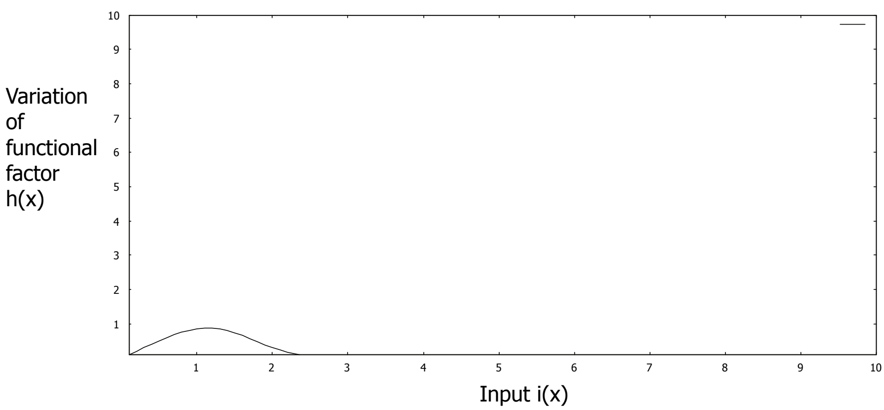

Fig. 15. Profile of network characteristic function for damped periodic input excitation.

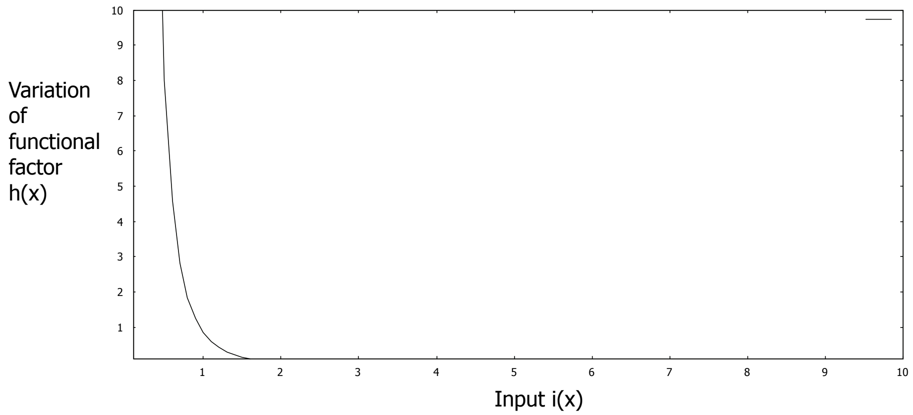

Fig. 16. Profile of network characteristic function for undamped periodic input excitation.

If the excitation to networks is damped periodic in nature, then the radius of convergence becomes narrow, as expected. However, a sharp convergent and non-linear profile is required for stability if the excitation to a non-stationary network is divergent and periodic in nature. The surface map of response of the dynamic networks under different input excitations is illustrated in Fig. 17. It is evident from Fig. 17 that, the response of the non-stationary networks is stable under various input excitation modes. The initial variations in responses are converged to a stable surface within a short interval.

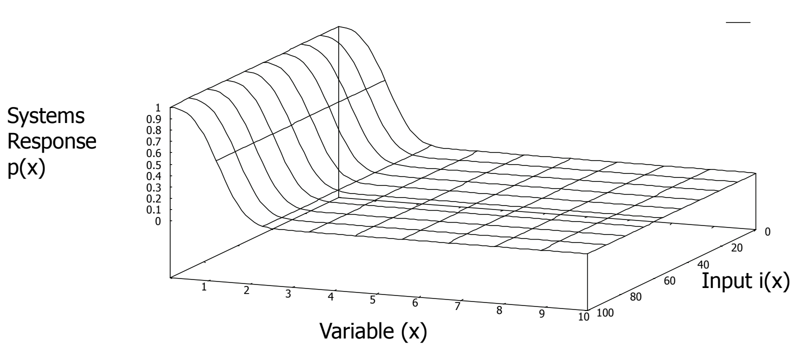

Fig. 17. Surface map of network response for divergent coefficient ratio.

5. Conclusion

The formulation of analytical solutions of ODE model of network systems is required to understand the dynamics of network under various excitations. However, the analysis of nonlinear ODE model of the network systems having forcing functions is difficult due to the existence of multivariate coupled coefficients. A decomposition of coefficients and appropriate linearization are needed to resolve the input excitations independent of any arbitrary external function. Furthermore, the analytical solution under decomposition provides an insight to the stability property of the network systems. The analysis exposes required network characteristic functions of different natures in order to maintain stability of networks.

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